Approximation of infima in the calculus of variations

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Abstract

The goal of this paper is to give numerical estimates for some problems of the Calculus of Variations in the nonhomogeneous scalar case. The stored energy function considered is then a function $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$. We try to compare the infimum of the energy defined by $\varphi$ on a Sobolev space, with the infimum of the same energy on a finite element space, in terms of the mesh size. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $n \geq 1$ be an integer, $\Omega$ be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary and

$\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$

be a continuous function. Let us define $\overline{\varphi} : W^{1,\infty}(\Omega) \rightarrow \mathbb{R}$ by

$\overline{\varphi}(u) = \inf_{w \in u + W^{1,\infty}_0(\Omega)} \frac{1}{|\Omega|} \int_{\Omega} \varphi(x, \nabla w(x)) \, dx.$

Usually, in various problems of the Calculus of Variations, one is interested in the existence of a function $v \in W^{1,\infty}(\Omega)$ such that

$\overline{\varphi}(u) = \frac{1}{|\Omega|} \int_{\Omega} \varphi(x, \nabla v(x)) \, dx$ \quad and \quad $v = u$ \quad on $\partial \Omega$.

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When we look at $\bar{\varphi}(u)$, we look at a minimization problem where $u$ is a boundary data. But the fact that $\varphi$ is depending on $x$ allows us to include the boundary data in the function $\varphi$; more precisely, if we consider the minimization problem

$$\inf_{w \in W^{1,\infty}_0(\Omega)} \frac{1}{|\Omega|} \int_{\Omega} \varphi(x, \nabla w(x)) \, dx$$

and if we set $\varphi_u(x, \beta) = \varphi(x, \nabla u(x) + \beta)$ we get the following expression for (1.1):

$$\inf_{w \in W^{1,\infty}_0(\Omega)} \frac{1}{|\Omega|} \int_{\Omega} \varphi_u(x, \nabla w(x)) \, dx.$$  

This is the reason why we will just consider in the following the quantity $\bar{\varphi}(0)$.

The aim of this paper is to show that we can approach $\bar{\varphi}(0)$ by an infimum where the Sobolev space $W^{1,\infty}_0(\Omega)$ is replaced by a $P_1$-finite element space $V_h(\Omega)$. We will give more precisely estimates for the difference $\bar{\varphi}^e(0) - \bar{\varphi}(0)$.

Such numerical estimates for problems of the Calculus of Variations — especially for some of which concerning material science and elastic crystals — were considered by many authors; see, for example, [2–14, 17–20].

We will consider the following hypothesis:

(H$_1$) $\forall x \in \Omega, \forall \beta \in \mathbb{R}^n, a_1(x) + b_1 |\beta|^p \leq \varphi(x, \beta) \leq a_2(x) + b_2 |\beta|^p$,

(H$_2$) $\forall x, y \in \Omega, \forall \beta \in \mathbb{R}^n$, $|\varphi(x, \beta) - \varphi(y, \beta)| \leq \theta_1|x - y|(1 + b_2 |\beta|^p)$,

(H$_3$) $\forall x \in \Omega, \forall \beta_1, \beta_2 \in \mathbb{R}^n$, $|\varphi(x, \beta_1) - \varphi(x, \beta_2)| \leq \theta_2 |\beta_1 - \beta_2|((\theta_3 + (|\beta_1| + |\beta_2|)^p)$,

where $a_1, a_2 \in L^\infty(\Omega)$, $b_2 > b_1 > 0$, $p > 1$ and $\theta_1, \theta_2, \theta_3 > 0$.

Of course, if $\varphi$ satisfies some of the assumptions (H$_1$)–(H$_3$), then $\varphi_u$ does also provided $u$ has some smoothness. See Remark 1.5 below.

**Remark 1.1.** By definition of $\bar{\varphi}$ it is clear that $\forall u \in W^{1,\infty}(\Omega)$ and $\forall v \in W^{1,\infty}_0(\Omega)$, one has

$$\bar{\varphi}(u + v) = \bar{\varphi}(u).$$

So, $\bar{\varphi}(u)$ depends only on the boundary values of $u$.

**Remark 1.2.** Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ such that $u(x) = a \cdot x + b$.

Assume

$$\forall x \in \Omega, \forall \beta \in \mathbb{R}^n, \varphi(x, \beta) = \psi(\beta),$$

then $\bar{\varphi}(u) = \psi^{**}(a)$, where $\psi^{**}$ denotes the convex envelope of $\psi$, see [15].
Remark 1.3. Let us denote by $\varphi^{**}$ the map of $\Omega \times \mathbb{R}^n$ into $\mathbb{R}$ such that if $x \in \Omega$, $\varphi^{**}(x, \cdot)$ is the convex envelope of $\varphi(x, \cdot)$.

- If (H$_1$) holds for $\varphi$, then it holds for $\varphi^{**}$ also.

Indeed, it follows from the convexity of the function $\beta \mapsto a_1(x) + b_1 |\beta|^p$.

- If (H$_1$) and (H$_2$) hold for $\varphi$, then for $x, y, \in \Omega$ fixed, we have that

$$\forall \beta \in \mathbb{R}^n, \quad \varphi(x, \beta) \leq \varphi(y, \beta) + \theta_1 |x - y| (1 + b_2 |\beta|^p)$$

$$\leq \varphi(y, \beta) + \theta_1 |x - y| \left( 1 + \frac{b_2}{b_1} (\varphi(y, \beta) - a_1(y)) \right)$$

$$\leq \left( 1 + \frac{b_2}{b_1} \theta_1 |x - y| \right) \varphi(y, \beta) + \theta_1 |x - y| \left( 1 - \frac{b_2}{b_1} a_1(y) \right)$$

and, thus,

$$\forall \beta \in \mathbb{R}^n, \quad \varphi^{**}(x, \beta) \leq \left( 1 + \frac{b_2}{b_1} \theta_1 |x - y| \right) \varphi^{**}(y, \beta) + \theta_1 |x - y| \left( 1 - \frac{b_2}{b_1} a_1(y) \right).$$

In particular, we get

$$\varphi^{**}(x, 0) - \varphi^{**}(y, 0) \leq \theta_1 |x - y| \left( 1 + \frac{b_2}{b_1} (a_2(y) - a_1(y)) \right)$$

$$\leq \theta'_1 |x - y|$$

with

$$\theta'_1 = \theta_1 \left( 1 + \frac{b_2}{b_1} (|a_1|_{L^\infty} + |a_2|_{L^\infty}) \right).$$

Reversing $x$ and $y$ we obtain

$$|\varphi^{**}(x, 0) - \varphi^{**}(y, 0)| \leq \theta'_1 |x - y|. \quad (1.2)$$

Remark 1.4. If $\varphi$ is such that (H$_1$) and (H$_2$) hold, then

$$\bar{\varphi} = \varphi^{**}$$

("Relaxation theorem": see [15], Corollary 2.2, Chapter 5, Section 5.2, p. 235).

Remark 1.5. Let us consider $u \in W^{1, \infty}(\Omega)$ and define as above $\varphi_u(x, \beta) = \varphi(x, \nabla u(x) + \beta)$. If $\nabla u$ is continuous then $\varphi_u$ is continuous when $\varphi$ is. Moreover,

- If $u \in W^{1, \infty}(\Omega)$ and if (H$_1$) holds for $\varphi$, then (H$_1$) holds for $\varphi_u$ with some other functions $a'_1$ and $a'_2$ (depending on $u$) and some other constants $b'_1$ and $b'_2$.

Indeed

$$\varphi_u(x, \beta) = \varphi(x, \nabla u(x) + \beta)$$

$$\geq a_1(x) + b_1 |\nabla u(x) + \beta|^p$$

$$\geq a_1(x) - b_1 |\nabla u(x)|^p + 2^{1-p} b_1 |\beta|^p$$
and
\[
\phi_u(x, \beta) \leq a_2(x) + b_2|\nabla u(x) + \beta|^p \\
\leq a_2(x) + 2^{p-1}b_2|\nabla u(x)|^p + 2^{p-1}b_2|\beta|^p.
\]

- If \( u \in W^{2,\infty}(\Omega) \), if (H2) and (H3) hold for \( \phi \), then (H2) holds for \( \phi_u \) with some other constants \( \theta_1'' \) and \( b_2'' \) (depending on \( u \)).

In fact, it follows from (H2) and (H3) that
\[
|\phi_u(x, \beta) - \phi_u(y, \beta)| \\
= |\phi(x, \nabla u(x) + \beta) - \phi(y, \nabla u(y) + \beta)| \\
\leq |\phi(x, \nabla u(x) + \beta) - \phi(x, \nabla u(y) + \beta)| + |\phi(x, \nabla u(y) + \beta) - \phi(y, \nabla u(y) + \beta)| \\
\leq \theta_2|\nabla u(x) - \nabla u(y)| \{\theta_3 + (2|\beta| + |\nabla u(x)| + |\nabla u(y)|)^p\} \\
+ \theta_1|x - y|(1 + b_2|\nabla u(y) + \beta|^p) \\
\leq \theta_2C_u|x - y| \{\theta_3 + 2^{p-1}(|\nabla u(x)| + |\nabla u(y)|)^p + 2^{p-1}|\beta|^p\} \\
+ \theta_1|x - y|(1 + 2^{p-1}b_2|\nabla u(y)|^p + 2^{p-1}b_2|\beta|^p) \\
\leq \theta_1''|x - y|(1 + b_2''|\beta|^p).
\]

- Finally, let us prove that
\[
\forall x \in \Omega, \forall \beta \in \mathbb{R}^n, \quad (\phi_u)^*(x, \beta) = \phi^*(x, \nabla u(x) + \beta).
\] (1.3)

Indeed, let \( x \in \Omega \) be fixed. The function \( \beta \mapsto \phi^*(x, \nabla u(x) + \beta) \) is convex and verifies
\[
\forall \beta \in \mathbb{R}^n, \quad \phi^*(x, \nabla u(x) + \beta) \leq \phi(x, \nabla u(x) + \beta) = \phi_u(x, \beta)
\]
implying that
\[
\phi^*(x, \nabla u(x) + \cdot) \leq (\phi_u)^*(x, \cdot).
\]

Conversely, the function \( \beta \mapsto (\phi_u)^*(x, \beta - \nabla u(x)) \) is convex and verifies
\[
\forall \beta \in \mathbb{R}^n, \quad (\phi_u)^*(x, \beta - \nabla u(x)) \leq \phi_u(x, \beta - \nabla u(x)) = \phi(x, \beta)
\]
implying that
\[
\forall \beta \in \mathbb{R}^n, \quad (\phi_u)^*(x, \beta - \nabla u(x)) \leq \phi^*(x, \beta)
\]
and thus
\[
\forall \beta \in \mathbb{R}^n, \quad (\phi_u)^*(x, \beta) \leq \phi^*(x, \nabla u(x) + \beta).
\]

So (1.3) is proved.

The following result gives some useful properties of \( \overline{\phi} \):
Theorem 1.1. If \( \phi \) satisfies (H₁) and (H₂), then the function \( \overline{\phi} : W^{1,\infty}(\Omega) \to \mathbb{R} \) is convex and locally Lipschitz.

Proof. First, let us prove that \( \overline{\phi} \) is convex.

Let \( u, v \in W^{1,\infty}(\Omega) \) and \( \varepsilon > 0 \). By definition of \( \overline{\phi} \) and since \( \overline{\phi} = \phi^{**} \) (see Remark 1.5), there exist \( u_\varepsilon, v_\varepsilon \in W^{1,\infty}(\Omega) \) such that

\[
\overline{\phi}(u) \geq \frac{1}{|\Omega|} \int_\Omega \phi^{**}(x, \nabla u(x) + \nabla u_\varepsilon(x)) \, dx - \varepsilon
\]

and

\[
\overline{\phi}(v) \geq \frac{1}{|\Omega|} \int_\Omega \phi^{**}(x, \nabla v(x) + \nabla v_\varepsilon(x)) \, dx - \varepsilon.
\]

Then, since \( \phi^{**} \) is convex, one has for \( \lambda \in [0, 1] \)

\[
\lambda \overline{\phi}(u) + (1 - \lambda) \overline{\phi}(v) \geq \frac{1}{|\Omega|} \int_\Omega [\lambda \phi^{**}(x, \nabla u(x) + \nabla u_\varepsilon(x))
+ (1 - \lambda) \phi^{**}(x, \nabla v(x) + \nabla v_\varepsilon(x))] \, dx - \varepsilon
\]

\[
\geq \frac{1}{|\Omega|} \int_\Omega \phi^{**}(x, \lambda \nabla u(x) + (1 - \lambda) \nabla v(x) + \nabla w_\varepsilon(x)) \, dx - \varepsilon
\]

where \( w_\varepsilon = \lambda u_\varepsilon + (1 - \lambda) v_\varepsilon \in W^{1,\infty}_0(\Omega) \).

Thus,

\[
\lambda \overline{\phi}(u) + (1 - \lambda) \overline{\phi}(v) \geq \overline{\phi}(\lambda u + (1 - \lambda) v) - \varepsilon
\]

and this inequality being true for all \( \varepsilon > 0 \), we obtain the convexity of \( \overline{\phi} \).

Now, let us prove that \( \overline{\phi} \) is locally Lipschitz. For all function \( v \in W^{1,\infty}_0(\Omega) \) the map

\[
\psi_v : u \mapsto \frac{1}{|\Omega|} \int_\Omega \phi(x, \nabla u(x) + \nabla v(x)) \, dx
\]

is continuous on \( W^{1,\infty}(\Omega) \), in such way that

\[
\overline{\phi} = \inf_{v \in W^{1,\infty}_0(\Omega)} \psi_v
\]

is upper semicontinuous, and so \( \forall u \in W^{1,\infty}(\Omega) \) there exist a neighbourhood of \( u \) where \( \overline{\phi} \) is bounded from above. Consequently, since \( \overline{\phi} \) is convex, we deduce that \( \overline{\phi} \) is locally Lipschitz on \( W^{1,\infty}(\Omega) \), (see [16] Corollary 2.4, Chapter I, p. 12). \( \square \)
2. Approximation

From now on, we will assume that $\Omega$ is a polygonal domain. Then, let $\{\mathcal{T}_h; h > 0\}$ be a family of regular triangulation of $\Omega$ (see [21]), that is to say satisfying

\[
\forall h > 0 \quad \begin{cases}
\forall K \in \mathcal{T}_h, & K \text{ is a } n\text{-simplex}, \\
\max_{K \in \mathcal{T}_h} (h_K) = h, \\
\forall K \in \mathcal{T}_h, & \frac{h_K}{\rho_K} \leq v \quad (v > 0),
\end{cases}
\]

where $h_K$ is the diameter of the $n$-simplex $K$ and $\rho_K$ its roundness (i.e. the largest diameter of the balls that could fit into $K$).

If $P_1(K)$ is the space of polynomials of degree 1 on $K$, we set

\[V^h(\Omega) = \{ v : \Omega \to \mathbb{R} \text{ continuous: } v|_K \in P_1(K), \forall K \in \mathcal{T}_h \}\]

and

\[V_0^h(\Omega) = \{ v \in V^h(\Omega): v = 0 \text{ on } \partial \Omega \}.\]

Then, we define $\overline{\varphi}^h : W^{1,\infty}(\Omega) \to \mathbb{R}$ by

\[\forall u \in W^{1,\infty}(\Omega), \quad \overline{\varphi}^h(u) = \inf_{w \in u + V_0^h(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(x, \nabla w(x)) \, dx.\]

Now, our goal is to get estimates of $\overline{\varphi}^h(u) - \overline{\varphi}(u)$ (we know that $\varphi^h(u) \geq \overline{\varphi}(u)$ since $W_0^{1,\infty}(\Omega)$ contains $V_0^h(\Omega)$) and in particular of $\varphi^h(0) - \overline{\varphi}(0)$; see Section 1.

**Remark 2.1.** It follows from (2.1) that, if $v \in W^{1,\infty}(\Omega)$, the function $v_h \in V^h(\Omega)$, that interpolates $v$ on $\mathcal{T}_h$, satisfies

\[|\nabla v_h(x)| \leq C_0 |\nabla v(x)| \quad \text{a.e. in } \Omega\]

for some constant $C_0$ depending only on $n$ and $v$. See [4], Lemma 2.1.

**Remark 2.2.** If $u \in W^{1,\infty}(\Omega)$ and $v \in V_0^h(\Omega)$, one has $\overline{\varphi}^h(u + v) = \overline{\varphi}^h(u)$.

**Remark 2.3.** If $\forall x \in \Omega, \forall \beta \in \mathbb{R}^n, \varphi(x, \beta) = \psi(\beta)$, and $u(x) = a \cdot x + b$ ($a \in \mathbb{R}^n$, $b \in \mathbb{R}$), see Remark 1.2, one will find estimates of $\overline{\varphi}^h(u) - \overline{\varphi}(u)$ in [2–4].
3. A preliminary estimate

For a set $A \subset \mathbb{R}^n$, we denote by $co(A)$ the convex hull of $A$.

In this part, we will use some "saw-tooth" function constructed in [5]; let us recall that:

**Lemma 3.1.** Let $\omega_0, \ldots, \omega_k \in \mathbb{R}^n$ be such that the dimension of the vector space $W$ spanned by $\omega_1 - \omega_0, \ldots, \omega_k - \omega_0$ is equal to $k$ and such that $0 \in co\{\omega_0, \ldots, \omega_k\}$. For each $\eta > 0$ there exists a piecewise affine function $w : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$|w(x)| \leq \eta \quad \forall x \in \mathbb{R}^n,$$

$$\nabla w(x) \in \{\omega_0, \ldots, \omega_k\} \quad a.e. \ in \ \mathbb{R}^n$$

and if we denote by $S$ the subset of $\mathbb{R}^n$ where $w$ has no derivative, then, for any bounded domain $D \subset \mathbb{R}^n$, we have that

the $(n - 1)$-dimensional measure of $D \cap S$ is less than $C_1|D|\eta^{-1}$, \hspace{1cm} (3.3)

where $C_1$ is a constant only depending on $\omega_0, \ldots, \omega_k$.

**Proof.** See [5], especially the proof of Theorem 1.

**Theorem 3.1.** Let $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying

- $\psi$ is nonnegative
- $\psi(0, \cdot)$ is bounded on any bounded sets of $\mathbb{R}^n$
- for almost every $x \in \Omega$, $\exists \omega_0(x), \ldots, \omega_n(x)$ (not necessarily two by two distinct) such that
  
  (i) $0 \in co\{\omega_0(x), \ldots, \omega_n(x)\}$,
  (ii) $\forall i = 0, \ldots, n \ \psi(x, \omega_i(x)) = 0$,
  (iii) $\exists K > 0$ such that for a.e. $x \in \Omega$, $\forall i = 0, \ldots, n \ |\omega_i(x)| \leq K$

- there exist $\kappa_1, \kappa_2 \geq 0$ and $p > 1$ such that $\forall \beta \in \mathbb{R}^n$, $\forall x, y \in \Omega$ one has

$$|\psi(x, \beta) - \psi(y, \beta)| \leq \kappa_1 |x - y|(1 + \kappa_2 |\beta|^p).$$

Then $\forall h > 0$, $\exists u_h \in V^h_0(\Omega)$ such that

$$|u_h|_{L^\infty(\Omega)} \leq h^{2/3}$$

and

$$\frac{1}{|\Omega|} \int_\Omega \psi(x, \nabla u_h(x)) \, dx \leq Ch^{1/3},$$

where $C$ is a constant depending on $\psi, \Omega, K, \kappa_1, \kappa_2$ and $p$.

In particular, $0 \leq \bar{\psi}^h(0) \leq Ch^{1/3}$.

**Proof.** Let us denote by $\gamma$ and $\delta$ some real numbers such that $0 < \gamma < \delta < 1$ (remark that for $h \ll 1$ we have $h \ll h^\gamma \ll h^\delta$).
Let us cover $\Omega$ by $n$-cubes $Q_1, \ldots, Q_s$, $Q'_1, \ldots, Q'_r$ of side $h^i$ where

$$\forall i = 1, \ldots, s, \quad Q_i \subset \Omega \quad \text{and} \quad \forall j = 1, \ldots, r, \quad Q'_j \cap \partial \Omega \neq \emptyset.$$ 

For each $i \in \{1, \ldots, s\}$ let us consider $x_i \in Q_i$ satisfying (3.5). Without loss of generality, we can assume, if the dimension of the vector space $W_i$ spanned by $\omega_i(x_i) - \omega_0(x_i), \ldots, \omega_n(x_i) - \omega_0(x_i)$ is equal to $k_i$, that $\omega_i(x_i) - \omega_0(x_i), \ldots, \omega_n(x_i) - \omega_0(x_i)$ are linearly independent.

Now, let us consider the function $w_i$ of the Lemma 3.1 corresponding to $\eta = h^\delta$ and $D = Q_i$, and denote by $v_i$ the function defined on $Q_i$ by

$$x \mapsto \max\{\min(d(x, \partial Q_i); w_i(x)); -d(x, \partial Q_i)\}.$$ 

Since $v_i = 0$ on $\partial Q_i$, we can define on $\Omega$ a continuous function $u$ equal to $v_i$ on $Q_i$ and to 0 on $Q'_j$. Then, let us denote by $u_h$ the $\mathcal{T}_h$-interpolate of $u$.

First, $u_h \in V^h_0(\Omega)$. Secondly, it follows from the definition of $u$ and (3.1) that

$$|u_h(x)| \leq h^\delta \quad \text{a.e. in } \Omega. \quad (3.9)$$

Third, by (3.1), (3.2), (2.2) and (3.5) (iii) we get that

$$|\nabla u_h(x)| \leq C_0 |\nabla u(x)| \leq C_2 \quad \text{a.e. in } \Omega. \quad (3.10)$$

To obtain an estimate of

$$\int_{\Omega} \psi(x, \nabla u_h(x)) \, dx$$

let us write

$$\int_{\Omega} \psi(x, \nabla u_h(x)) \, dx \leq \sum_{i=1}^s \int_{Q_i} |\psi(x, \nabla u_h(x)) - \psi(x_i, \nabla u_h(x))| \, dx$$

$$+ \sum_{i=1}^s \int_{Q_i} \psi(x_i, \nabla u_h(x)) \, dx + \int_{\Omega_{\partial}} \psi(x, \nabla u_h(x)) \, dx, \quad (3.11)$$

where $\Omega_{\partial} = \Omega \cap (Q'_1 \cup \cdots \cup Q'_r)$, and consider separately each of the terms in the right-hand side of this inequality.

• By using (3.6) and (3.10) we have that

$$\sum_{i=1}^s \int_{Q_i} |\psi(x, \nabla u_h(x)) - \psi(x_i, \nabla u_h(x))| \, dx \leq \sum_{i=1}^s \int_{Q_i} \kappa_1 |x - x_i| (1 + \kappa_2 |\nabla u_h(x)|^\rho) \, dx$$

$$\leq \kappa_1 h^i (1 + \kappa_2 C^\rho_2) \sum_{i=1}^s |Q_i|$$

$$\leq C_3 |\Omega| h^\gamma. \quad (3.12)$$

• Since $|\Omega_{\partial}| \leq C_4 |\partial \Omega| h^i$, we have from (3.4), (3.6) and (3.10) that

$$\int_{\Omega_{\partial}} \psi(x, \nabla u_h(x)) \, dx \leq C_5 h^\gamma. \quad (3.13)$$
For each $i \in \{1, \ldots, s\}$, we have that
\[ \nabla u_h(x) \in \{ \omega_0(x), \ldots, \omega_h(x) \} \quad \text{a.e. in } Q_i \]
extcept, perhaps, on a neighbourhood $S_{i,1}$ of $\partial Q_i$ such that
\[ |S_{i,1}| \leq C_6 |\partial Q_i| (h^\delta + h) \leq C_7 |Q_i| h^{\delta - \gamma} \tag{3.14} \]
and on the set $S_{i,2}$ made on the $n$-simplices of $\mathcal{F}_h$ which intersect $S_i \cap Q_i$ (with, as in the Lemma 3.1, $S_i$ the subset of $\mathbb{R}^n$ where $w_i$ has no derivative); by definition of $S_{i,2}$ and (3.3) we have that
\[ |S_{i,2}| \leq C_8 \frac{|Q_i|}{h^\delta} \cdot h = C_8 |Q_i| h^{1-\delta}. \tag{3.15} \]
Hence, we have from (3.4), (3.6), (3.10), (3.14) and (3.15) that
\[ \sum_{i=1}^s \int_{Q_i} \psi(x, \nabla u_h(x)) \, dx = \sum_{i=1}^s \int_{S_{i,1} \cup S_{i,2}} \psi(x, \nabla u_h(x)) \, dx \]
\[ \leq C_9 \sum_{i=1}^s (|S_{i,1}| + |S_{i,2}|) \]
\[ \leq C_9 \left( \sum_{i=1}^s |Q_i| \right) (C_7 h^{\delta - \gamma} + C_8 h^{1-\delta}) \]
\[ \leq C_{10} |Q_i| (h^{\delta - \gamma} + h^{1-\delta}). \tag{3.16} \]
Finally, using (3.11)–(3.13) and (3.16) we get
\[ \int_{Q} \psi(x, \nabla u_h(x)) \, dx \leq C_{11} |Q_i| (h^\gamma + h^{\delta - \gamma} + h^{1-\delta}) \]
\[ \leq 3C_{11} |Q_i| h^{\min(\gamma, \delta - \gamma, 1-\gamma)}. \]
So, by choosing $\gamma = \frac{1}{3}$ and $\delta = \frac{2}{3}$, we obtain (3.8), and (3.9) give (3.7). \(\square\)

**Remark 3.1.** To compute numerically
\[ \int_{Q} \psi(x, \nabla u_h(x)) \, dx \]
it seems to be natural to consider
\[ \sum_{K \in \mathcal{F}_h} |K| \psi(x_K, \beta_K) \]
where $\beta_K = \nabla u_h|_K$ and $x_K$ is the center of $K$. If $\psi$ satisfies (3.6), we have that
\[ \left| \int_{Q} \psi(x, \nabla u_h(x)) \, dx - \sum_{K \in \mathcal{F}_h} |K| \psi(x_K, \beta_K) \right| \leq C h. \]
4. Estimate of $\bar{\varphi}^h(0) - \bar{\varphi}(0)$

First, let us recall a geometrical lemma:

**Lemma 4.1.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that

$$
\lim_{|\beta| \to +\infty} \frac{f(\beta)}{|\beta|} = +\infty.
$$

Let $x \in \mathbb{R}^n$. Then, there exist $x_0, \ldots, x_n \in (f - f^{**})^{-1}(0)$ not necessarily two by two distinct such that

$$
x \in \text{co}\{x_0, \ldots, x_n\} \quad \text{and} \quad f^{**} \text{ is affine on co}\{x_0, \ldots, x_n\}
$$

(co$\{x_0, \ldots, x_n\}$ could be reduced to $\{x\}$).

**Proof.** See [4].

**Theorem 4.1.** Let us assume that $\varphi$ verifies $(H_1)$ and $(H_2)$ and that

$$
\bar{\varphi}(0) = \inf_{w \in W^1_0(\Omega)} \frac{1}{|\Omega|} \int_{\Omega} \varphi(x, \nabla w(x)) \, dx = \frac{1}{|\Omega|} \int_{\Omega} \varphi^{**}(x, 0) \, dx. \tag{4.1}
$$

Moreover, suppose that $\forall x \in \Omega, \exists A(x) \in \partial \varphi^{**}(x, 0)$ verifying for some constant $C_1$

$$
|A(x) - A(y)| \leq C_1 |x - y|, \quad \forall x, y \in \Omega. \tag{4.2}
$$

Then, $\forall h > 0,$

$$
0 \leq \bar{\varphi}^h(0) - \bar{\varphi}(0) \leq Ch^{1/3},
$$

where $C$ is a constant depending on $\varphi, \Omega, C_1$ and the constants appearing in $(H_1)$ and $(H_2)$.

**Proof.** (Let us recall that $\partial \varphi^{**}(x, 0)$ denotes the subdifferential of $\varphi^{**}(x, \cdot)$ at the point 0). Let $x \in \Omega$. Owing to $(H_1)$, one has

$$
\lim_{|\beta| \to +\infty} \frac{\varphi(x, \beta)}{|\beta|} = +\infty.
$$

Thus, from the continuity of $\varphi$ and Lemma 4.1, there exist $x_0(x), \ldots, x_n(x) \in \mathbb{R}^n$ (not necessarily pairwise distinct) such that

$$
\forall i = 0, \ldots, n \quad \varphi(x, x_i(x)) = \varphi^{**}(x, x_i(x)) \tag{4.3}
$$

$$
0 \in \text{co}\{x_0(x), \ldots, x_n(x)\} \tag{4.4}
$$

$$
\varphi^{**}(x, \cdot) \text{ is affine on co}\{x_0(x), \ldots, x_n(x)\}. \tag{4.5}
$$
Next, set
\[ g : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}, \]
\[ (x, \beta) \mapsto A(x) \cdot \beta + \varphi(x,0). \]

Let us remark that we have, in particular,
\[ g(x,0) = \varphi(x,0). \tag{4.6} \]

Now, we are going to prove that one can use the Theorem 3.1 for the function \( \psi = \varphi - g. \)

- First, \( \psi \geq 0; \) indeed \( \forall x \in \Omega, \forall \beta \in \mathbb{R}^n \) one has
  \[ \psi(x,\beta) = \varphi(x,\beta) - g(x,\beta) \]
  \[ \geq \varphi(x,\beta) - \varphi(x,0) \]
  since \( A(x) \in \partial \varphi(x,0) \) means that
  \[ \forall \beta \in \mathbb{R}^n, \quad A(x) \cdot \beta \leq \varphi(x,\beta) - \varphi(x,0). \tag{4.7} \]

  Thus \( \psi(x,\beta) \geq 0. \)

- Second, since \( \varphi \) and \( g(0,\cdot) \) are continuous, then \( \psi \) satisfies (3.4).

- Third, \( \forall x \in \Omega, \) one has (4.4) and \( \psi(x,\alpha_i(x)) = \varphi(x,\alpha_i(x)) - g(x,\alpha_i(x)); \) but from (4.4) and (4.5) we deduce that \( g(x,\cdot) \) and \( \psi(x,\cdot) \) coincide on \( co\{\alpha_0(x),\ldots,\alpha_n(x)\} \) (indeed, from (4.7) one has \( \varphi(x,\cdot) \geq g(x,\cdot) \) and if \( g(x,\alpha_j(x)) < \varphi(x,\alpha_j(x)) \) for some \( j, \) then
  \[ g(x,0) = \sum_{i=0}^n \lambda_i g(x,\alpha_i(x)) < \sum_{i=0}^n \lambda_i \varphi(x,\alpha_i(x)) = \varphi(x,0) \]
  and this is a contradiction with (4.6). Note that, without loss of generality, we can assume that \( \lambda_i > 0. \)

  Consequently, by (4.3) we have that
  \[ \psi(x,\alpha_i(x)) = \varphi(x,\alpha_i(x)) - \varphi(x,\alpha_i(x)) = 0. \]

- Fourth, \( \forall x, y \in \Omega, \forall \beta \in \mathbb{R}^n \) one has that
  \[ |\psi(x,\beta) - \psi(y,\beta)| \leq |\varphi(x,\beta) - \varphi(y,\beta)| + |g(x,\beta) - g(y,\beta)| \]
  \[ \leq \theta_1 |x - y| (1 + b_2 |\beta|^p) + |A(x) - A(y)| |\beta| + |\varphi(x,0) - \varphi(y,0)|. \]

  By using (4.2) and (1.2), we get
  \[ |\psi(x,\beta) - \psi(y,\beta)| \leq |x - y| (\theta_1 (1 + b_2 |\beta|^p) + C_1 |\beta| + \theta_1') \]
  \[ \leq |x - y| ((\theta_1 + C_1 + \theta_1') (b_2 + C_1) |\beta|^p). \]

- Finally, it remains to show that there exists a constant \( K > 0 \) independent of \( x, \) such that
  \[ \forall i = 0, \ldots, n, \quad \forall x \in \Omega, \quad |\alpha_i(x)| \leq K. \tag{4.8} \]
We know that $\forall x \in \Omega$, $\forall \beta \in \mathbb{R}^n$, one has from (H1) and the Remark 1.3
\[
a_1(x) + b_1|\beta|^p \leq \phi^{**}(x, \beta) \leq a_2(x) + b_2|\beta|^p
\]
thus, for $\beta \neq 0$,
\[
\frac{a_1(x)}{|\beta|} + b_1|\beta|^{p-1} \leq \frac{\phi^{**}(x, \beta)}{|\beta|}.
\]
Therefore,
\[
\frac{\phi^{**}(x, \beta)}{|\beta|} \geq b_1|\beta|^{p-1} - \frac{|a_1|_{L^\infty(\Omega)}}{|\beta|}
\]
and
\[
\frac{\phi^{**}(x, \beta) - \phi^{**}(x, 0)}{|\beta|} \to +\infty \quad \text{uniformly with respect to } x, \text{ when } |\beta| \to +\infty.
\]

Since the function $A : \Omega \to \mathbb{R}^n$ is Lipschitz, and $\Omega$ bounded, one has $A \in L^\infty(\Omega, \mathbb{R}^n)$. Then, there exists $K > 0$ such that
\[
|\beta| > K \implies \phi^{**}(x, \beta) - \phi^{**}(x, 0) > |A|_{L^\infty(\Omega)} |\beta|, \quad \forall x \in \Omega.
\]  
(4.9)

Let us suppose that there exists $x \in \Omega$ and $i \in \{0, \ldots, n\}$ such that $|x_i(x)| > K$, then by (4.9) we have that
\[
A(x).\alpha_i(x) = \phi^{**}(x, \alpha_i(x)) - \phi^{**}(x, 0) > |A|_{L^\infty(\Omega)} |\alpha_i(x)|
\]  
(4.10)

but, using the Cauchy–Schwarz inequality, we get
\[
|A|_{L^\infty(\Omega)} |\alpha_i(x)| \geq |A(x)| |\alpha_i(x)| \geq |A(x).\alpha_i(x)|,
\]
thus (4.10) implies $A(x).\alpha_i(x) > |A(x).\alpha_i(x)|$ which is impossible.

Consequently (4.8) holds

Now, we can apply the Theorem 3.1. Therefore, $\forall h > 0, \exists u_h \in V_h^0(\Omega)$ such that
\[
|u_h|_{L^\infty(\Omega)} \leq h^{2/3} \quad \text{and} \quad \frac{1}{|\Omega|} \int_{\Omega} \psi(x, \nabla u_h(x)) \, dx \leq C_2 h^{1/3}
\]
(where $C_2$ is independent of $h$).

Then
\[
\tilde{\phi}^h(0) \leq \frac{1}{|\Omega|} \int_{\Omega} \psi(x, \nabla u_h(x)) \, dx + \frac{1}{|\Omega|} \int_{\Omega} g(x, \nabla u_h(x)) \, dx
\]
\[
\leq C_2 h^{1/3} + \frac{1}{|\Omega|} \int_{\Omega} A(x).\nabla u_h(x) \, dx + \frac{1}{|\Omega|} \int_{\Omega} \phi^{**}(x, 0) \, dx
\]
and, since the second integral in the last inequality is equal to $\tilde{\phi}(0)$, one has
\[
\tilde{\phi}^h(0) - \tilde{\phi}(0) \leq C_2 h^{1/3} + \frac{1}{|\Omega|} \int_{\Omega} A(x).\nabla u_h(x) \, dx.
\]  
(4.11)
Next, since $A = (A_i) : \Omega \to \mathbb{R}^n$ is Lipschitz, for all $\Omega' \subset \subset \Omega$, all $s \in \mathbb{R}^n$ such that $|s| < d(\Omega', \partial \Omega)$ and all $x \in \Omega'$, one has

$$|A_i(x + s) - A_i(x)| \leq C_1 |s|$$

and thus $A_i \in W^{1,\infty}(\Omega)$ and $|\nabla A_i|_{L^\infty(\Omega)} \leq C_1$ (see [1] Proposition IX.3, p. 153).

Therefore, we can write

$$\left| \int_\Omega A(x) \cdot \nabla u_h(x) \, dx \right| = \int_\Omega \text{div} A(x) u_h(x) \, dx \leq |u_h|_{L^\infty} \int_\Omega |\text{div} A(x)| \, dx \leq nC_1 |\Omega| h^{1/3}$$

and thanks to (4.11) we obtain

$$\bar{\varphi}^h(0) - \bar{\varphi}(0) \leq Ch^{1/3},$$

where $C$ is a constant depending on $\varphi, \Omega, C_1$ and on the constants from $(H_1)$ and $(H_2)$. Now the proof is complete. □

**Remark 4.1.** In the previous theorem, hypothesis (4.1) is satisfied, for instance, in the following case:

$$\forall \beta \in \mathbb{R}^n \quad \text{for a.e. } x \in \Omega, \quad \varphi(x, \beta) \geq \varphi^{**}(x, 0). \quad (4.12)$$

Indeed, if (4.12) holds, then for $v \in W^{1,\infty}_0(\Omega)$, we have that

$$\int_\Omega \varphi(x, \nabla v(x)) \, dx \geq \int_\Omega \varphi^{**}(x, 0) \, dx \geq \inf_w \int_\Omega \varphi^{**}(x, \nabla w(x)) \, dx = \inf_w \int_\Omega \varphi(x, \nabla w(x)) \, dx$$

and (4.1) follows. Moreover, for $\varphi$ having the symmetry property:

$$\forall \beta \in \mathbb{R}^n \quad \text{for a.e. } x \in \Omega, \quad \varphi(x, \beta) = \varphi(x, -\beta), \quad (4.13)$$

then (4.12) holds. In fact, we derive from (4.13) that

$$\varphi^{**}(x, 0) \leq \frac{1}{2} \varphi^{**}(x, \beta) + \frac{1}{2} \varphi^{**}(x, -\beta) \leq \varphi(x, \beta).$$

Hypothesis (4.2) holds, for instance, if $\forall x \in \Omega$, $\varphi^{**}(x, \cdot)$ has derivative at 0 and if the function $x \mapsto (\partial \varphi^{**}/\partial \beta)(x, 0)$ is Lipschitz.

**Theorem 4.2.** Let us assume that the function $\varphi$ satisfies $(H_1)$–$(H_3)$, and

$$\bar{\varphi}(0) = \inf_{w \in W^{1,\infty}_0(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(x, \nabla w(x)) \, dx = \frac{1}{|\Omega|} \int_\Omega \varphi^{**}(x, \nabla a(x)) \, dx, \quad (4.14)$$

where $a \in W^{2,\infty}_0(\Omega) \cap W^{1,\infty}(\Omega)$. 

Moreover, suppose that \( \forall x \in \Omega, \exists A(x) \in \partial \varphi^{**}(x, \nabla a(x)) \) verifying for some constant \( C_1 \)
\[ \forall x, y \in \Omega, \ |A(x) - A(y)| \leq C_1 |x - y| . \]
(4.15)

Then, \( \forall h > 0, \)
\[ 0 \leq \bar{\varphi}^h(0) - \bar{\varphi}(0) \leq C h^{1/3} , \]
where \( C \) is a constant depending on \( \varphi, \Omega, C_1 \), on the function \( a \) and on the constants from \( (H_1) - (H_3) \).

**Proof.** In order to use the previous theorem, let us introduce the following function \( \varphi_a : \Omega \times \mathbb{R}^n \to \mathbb{R} \)
defined by
\[ \forall x \in \Omega \text{ and } \forall \beta \in \mathbb{R}^n , \quad \varphi_a(x, \beta) = \varphi(x, \nabla a(x) + \beta) . \]

We would like to show that the function \( \varphi_a \) satisfies the assumptions of Theorem 4.1. First, we have that
\[ \bar{\varphi}_a(0) = \bar{\varphi}(a) \]
\[ = \bar{\varphi}(0) \text{ since } a \in W_0^{1, \infty} (\Omega) \quad \text{(see Remark 1.1)} \]
\[ = \frac{1}{|\Omega|} \int_{\Omega} \varphi^{**}(x, \nabla a(x)) \, dx \quad \text{by (4.14)} \]
\[ = \frac{1}{|\Omega|} \int_{\Omega} (\varphi_a)^{**}(x, 0) \, dx , \]
where the last equality results from (1.3).

Now, it is clear that \( \forall x \in \Omega, A(x) \in \partial (\varphi_a)^{**}(x, 0) \) and thus, by (4.15) we have (4.2) for the function \( \varphi_a \).

Finally, it remains to use the Remark 1.5 to see that \( (H_1) \) and \( (H_2) \) hold for \( \varphi_a \) (where the constant \( b_2 \) appearing in \( (H_1) \) and \( (H_2) \) is replaced by \( \max(b_2', b_2'') \)). To conclude, we apply Theorem 4.1; and this completes the proof. \( \square \)

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**References**